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Fully Nonlinear Ion-Acoustic Solitary Waves in a Magnetized Plasma

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The propagation of fully nonlinear ion-acoustic solitary waves in a magnetized plasma with cold ions and warm electrons is studied analytically. Necessary conditions for the existence of solitary waves in such a plasma were obtained by Yu *et al.* In this paper necessary and sufficient conditions are found.

1. INTRODUCTION

Propagation of ion-acoustic solitons in a magnetized plasma has been studied theoretically and experimentally by several authors and interesting results have been obtained which have physical applications. Through the derivation of a nonlinear wave equation Zakharov and Kuznetsov (1974) showed the existence of small-amplitude three-dimensional ion-acoustic solitons in a low-β magnetized plasma. In a kinetic approach, Swift (1975) obtained an equation for an oblique electrostatic shock in a magnetized plasma by solving the Poisson–Vlasov equation and calculating the ion density from adiabatic theory. Shukla and Yu (1978) showed that the finite-amplitude ion-acoustic soliton can occur in a plasma having an external magnetic field oblique to the propagating wave. Chaturvedi (1976) investigated the nonlinear electrostatic ion-cyclotron waves propagating nearly perpendicular to the magnetic field. Yu *et al.* (1980) investigated the fully nonlinear planar ion-acoustic solitary wave moving obliquely to an external magnetic field and obtained some conditions for the existence of a solitary wave in the plasma. On the other hand, a unified formulation was presented by Lee and Kan (1981) for the study of the nonlinear low-frequency electrostatic waves in a magnetized low-β plasma. Experiments

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on the propagation of an ion-acoustic soliton in a magnetized plasma have been described by Lonngren and co-workers (Hill *et al.*, 1987; Forsling *et al.*, 1989; Roychowdhury *et al.*, 1987), and important results have been obtained by them.

Theoretical investigation of plasma phenomena in a magnetized plasma, particularly the formation of an ion-acoustic soliton, has been carried out for the last few years, yet many problems on shocks, double layers, etc., have not been understood fully. In the present paper we investigate the fully nonlinear ion-acoustic solitary waves in a magnetized plasma having cold ions and warm electrons. We show that the conditions obtained by Yu *et al.* (1980) are not sufficient and some additional conditions are required for the existence of the ion-acoustic soliton in a magnetized plasma.

2. FORMULATION AND ANALYTICAL STUDY

For our present study, we start with the following equation obtained by Yu *et al.* (1980) for ion-acoustic solitary waves propagating in a magnetized plasma having cold ions and warm electrons:

$$\left(\frac{dn}{d\xi}\right)^2 + \psi(n) = 0 \tag{1}$$

where

$$\begin{aligned}\psi = \frac{n^4}{V^2(n^2 - V^2)} \{K_z^2 n^2 (n-1)^2 + 2nV^2[(1-K_z^2)n \ln n \\ -(n-K_z^2)(n-1)] + V^4(1-n)^2\}\end{aligned} \tag{2}$$

$$\xi = -Vt + K_x x + K_z Z, \qquad K_x^2 + K_z^2 = 1, \qquad n = n(\xi) \tag{3}$$

Here ψ is the Sagdeev potential, n is the ion density normalized with the background plasma density n_0 such that $n=1$ at $\xi = \pm\infty$, v is the speed of the localized pulse normalized with the ion-acoustic speed C_s, and K_x and K_z are the wave vectors in the x and z directions.

Yu *et al.* showed that equation (1) will have a soliton solution if

$$\psi(1) = C'(1) = 0 \tag{4}$$

$$\psi(N) = 0 \tag{5}$$

$$\psi(n) < 0 \tag{6}$$

for either $1 < n < N$ or $0 < N < n < 1$.

If $N>1$, then one has a density hump, and if $N<1$, one has a density cavity.

It is obvious that (5) and (6) are equivalent to the following:

(i) N is uniquely determined by

$$\psi(N)=0 \tag{7}$$

(ii) $\psi(n)<0$ near $n=1$ and $n=N$, i.e.,

$$\psi''(1)<0 \tag{8}$$

$$(N-1)\psi'(N)>0 \tag{9}$$

Yu *et al.* rightly noted that equation (2) satisfies equation (4) and they also checked the negativeness of ψ near $n=1$ and $n=N$. But they did not prove that there exists a unique N such that (7) is true. In order to check whether $\psi<0$ near $n=1$ and $n=N$, they expanded ψ in powers of $(n-1)$ and $(n-N)$ as

$$\psi=\frac{(V^2-K_z^2)(n-1)^2}{V^2(V^2-1)}-\frac{2(7K_z^2V^2-6V^4-3K_z^2+2V^2)}{3V^2(V^2-1)^2}(n-1)^3 \tag{10}$$

$$\psi=-\frac{N^3(N-1)(V^2-K_z^2N)}{V^2(N^2-V^2)}(n-N) \tag{11}$$

From (7)–(11), they obtained the condition

$$N^2>1>V^2 \tag{12}$$

Now, we shall check the negativeness of ψ near $n=1$ and $n=N$ and find whether N satisfying (7) is unique. Moreover, we shall see whether some restrictions on K_z, V, and N are needed to satisfy (8) and (9) from (2); we note that (8) is satisfied if and only if

$$K_z^2<V^2<1 \tag{13}$$

and (9) is satisfied if and only if

$$N>\frac{V^2}{K_z^2} \qquad \text{for} \quad 1<N<n \tag{14a}$$

$$N<V \qquad \text{for} \quad 0<N<n<1 \tag{14b}$$

The relation (14b) is not valid for the existence of the soliton solution, because we see from (2) that $\psi\to\infty$ and $n\to V$ and so for the soliton solution of (1) there exists an N ($<V$) such that $\psi(N)=0$ for $0<N<n<1$, which contradicts the result (14b). Consequently, $N<1$ is not possible.

Also note that one can get conditions (13) and (14) from the power series expansions of ψ near $n=1$ and $n=N$ given by (10) and (11), respectively. From (13) and (14), we get

$$N > \frac{V^2}{K_z^2} > 1 > V^2 \tag{15}$$

which includes the condition (12) found by Yu *et al.*

Our next task is to show that N (>1) satisfying (7) is unique. Equivalently we have to show that N (>1) is uniquely determined by the equation

$$\chi(n) = 0 \tag{16}$$

where

$$\chi(n) = \frac{V^2(n^2 - V^2)^2}{n^4}\,\psi(n) \tag{17}$$

Proof of uniqueness of N. From (17) we see that $\chi(1+\varepsilon) < 0$ and $\chi(\infty) > 0$, where ε (>0) is an arbitrary number. Consequently, $\eta(1+\varepsilon) < 0$ and $\eta(\infty) > 0$, where

$$\eta(n) = \frac{\chi(n)}{n^2} \tag{18}$$

Owing to the continuity of $\eta(n)$, there exists one N_1 such that

$$\eta(N_1) = 0 \qquad \text{and} \qquad N_1 > 1 \tag{19}$$

If possible, let there exist N_1 and N_2 such that

$$\chi(N_1) = \chi(N_2) = 0$$

i.e.,

$$\eta(N_1) = \eta(N_2) = 0 \qquad \text{and} \qquad N_2 > N_1 > 1 \tag{20}$$

From (20), applying Rolle's theorem, there exist N_3 and N_4 such that

$$\eta'(N_3) = \eta'(N_4) = 0 \qquad \text{and} \qquad N_2 > N_4 > N_1 > N_3 > 1 \tag{21}$$

But

$$\eta(n) = 2(n-1)F(n) \tag{22}$$

where

$$F(n)=K_z^2n^3-V^2n^2-V^2K_z^2n+V^4 \tag{23}$$

From (21)–(23),

$$F(N_3)=F(N_4)=0 \qquad \text{and} \qquad N_2>N_4>N_1>N_3>1 \tag{24}$$

But $F(n)$ is a polynomial of degree 3 such that $F(-\infty)<0$, $F(0)>0$, $F(1)<0$, and $F(\infty)>0$, and so we see that $F(n)$ vanishes only for $n>1$, which contradicts (24).

Hence there exists a unique N (>1) such that $\chi(N)=0$.

3. SUMMARY AND SOME CONCLUDING REMARKS

In this paper, the propagation of fully nonlinear ion-acoustic waves through a magnetoplasma has been studied and conditions for the existence of ion-acoustic solitons obtained. We have shown that a density hump of the solitary wave may be formed in a magnetoplasma, but a cavity would not be formed at all. From (15), we see that the conditions for a density hump are $N>(V^2/K_z^2)>1>V^2$, while the conditions for a hump obtained by Yu *et al.* (1980) are $N^2>1>V^2$. Further, we see that N is uniquely determined by $\psi(n)=0$ and $n\neq 0$.

The magnetoplasma of the earth's ionosphere is a finite-β plasma system. From satellite S3-3 observation, nonlinear ion-cyclotron waves have been observed along auroral magnetic field lines above the ionosphere (Temerin *et al.*, 1979). It is thought that nonlinear ion-cyclotron or ion-acoustic waves have an important role in the formation of small-scale auroral arcs (Sutradhar and Bujabarua, 1987). In the last few years several researchers have obtained the conditions for the existence of an ion-acoustic soliton in the earth's magnetosphere. But our results and conditions may be more suitable for the occurrence of solitons in the magnetosphere, in particular, the factor V^2/K_z^2 may have a vital role in the formation of a precursor of the solitary wave because this factor is related to the velocity of a localized pulse and the wave vector. Recently, Patel and Dasgupta (1987) reported some observational results on solitons in the earth's magnetosphere and explained theoretically the characteristics of solitary waves.

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REFERENCES

Chaturvedi, P. K. (1976). *Physics of Fluids*, **19**, 1064.
Forsling, P. S., Cooney, J., Tao, J., Resario, R., Lim, W. C., Seyhoonzadeh, A., and Lonngren, K. E. (1989). *Physica Scripta*, **40**, 188.
Hill, J., Roychowdhury, S., Chang, H. Y., Lien, C., Sukarto, S., and Lonngren, K. E. (1987). *Physica Scripta*, **36**, 503.
Lee, L. C., and Kan, J. R. (1981). *Physics of Fluids*, **24**, 430.
Patel, V. L., and Dasgupta, B. (1987). *Physica*, **27D**, 387.
Roychowdhury, S., Hill, J., Porsling, P. J., Chang, H. Y., Sukarto, S., Lien, C., and Lonngren, K. E. (1987). *Physica Scripta*, **36**, 508.
Shukla, P. K., and Yu, M. Y. (1978). *Journal of Mathematical Physics*, **16**, 2506.
Sutradhar, S., and Bujabarua, S. (1987). *Journal of the Physical Society of Japan*, **56**, 139.
Swift, D. (1975). *Journal of Geophysical Research*, **30**, 2096.
Temerin, M., Woldorff, M., and Mozer, F. S. (1979). *Physical Review Letters*, **43**, 1941.
Yu, M. L., Shukla, P. K., and Bujabarua, S. (1980). *Physics of Fluids*, **23**, 2146.
Zakharov, V. E., and Kuznetsov, E. A. (1974). *Zhurnal Eksp. i Teor. Fiz.* **66**, 594.
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